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HESSENBERG VARIETIES AND THEIR CELL DECOMPOSITIONS AND COHOMOLOGY RINGS

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This is a joint work with Takuro Abe, Tatsuya Horiguchi, Mikiya Masuda, and Satoshi Murai. In this note I introduce our results [3] and their interpretation from the point of view of hyperplane arrangements which are included in our paper implicitly.

1. Hessenberg varieties

Let G be a semisimple complex linear algebraic group of rank n, T a maximal torus of G, and B a Borel subgroup of G including T. A Hessenberg variety is a subvariety of the flag variety G/B which is determined by two data: one is an element of the Lie algebra $\mathrm{Lie}(G)$ and the other is a "good" subset of the positive root system Φ^+ of G, which is called a lower ideal. We found that Hessenberg varieties have very nice properties which the flag varieties also have.

Definition 1.1. For $\alpha, \beta \in \Phi^+$, we define $\alpha < \beta$ if $\beta - \alpha$ can be written as a nonnegative linear sum of the simple roots of G. A subset I of Φ^+ is a **lower ideal** if $\beta \in I$, $\alpha \in \Phi^+$, and $\alpha < \beta$ then $\alpha \in I$.

Definition 1.2. For $N \in \mathrm{Lie}(G)$ and a lower ideal I of Φ^+, the **Hessenberg variety** $\mathrm{Hess}(N, I)$ is the subvariety of G/B which is defined as follows:

$$\mathrm{Hess}(N, I) = \{gB \in G/B \mid \mathrm{Ad}(g^{-1})(N) \in \mathrm{Lie}(B) \oplus \bigoplus_{\alpha \in I} \mathfrak{g}_{-\alpha}\}.$$

When N is regular nilpotent (as a linear operator $[N, -]$ on $\mathrm{Lie}(G)$), we call $\mathrm{Hess}(N, I)$ a regular nilpotent Hessenberg variety. For two regular nilpotent elements N and N', the corresponding Hessenberg varieties $\mathrm{Hess}(N, I)$ and $\mathrm{Hess}(N', I)$ are isomorphic. Hence, in this note, we pay attention to how I affects the geometrical properties of regular nilpotent Hessenberg varieties.

2. Properties of regular nilpotent Hessenberg varieties

Before discussing properties of regular nilpotent Hessenberg varieties, let us recall some properties of flag varieies.

The flag variety G/B has the Bruhat decompotion $G/B = \coprod_{w \in W} C_w$, where C_w is the Schubert cell for w. The complex dimension of C_w is equal to the length of w, and it is well-known that the dimension is also equal to $\#N(w)$, where $N(w) = \{\alpha \in \Phi^+ \mid w^{-1}\alpha \in \Phi^-\}$. The maximal torus T acts on G/B by the left multiplication, and the Schubert cells are invariant under this action. We identify the Weyl group with the set of these cells and then the fixed point set. The rational cohomology ring of G/B is the quotient ring of $H^*(BT)$ by the ideal generated by the W-invariant elements [4]. Moreover the ideal is generated by a regular sequence of n elements when G is of rank n. By this fact (or the fact that flag varieties are orientable manifolds), the rational cohomology ring of the flag variety is a Poincaré duality algebra.

A regular nilpotent Hessenberg variety also has a natural cell decompotision. The cell decompotision of $\mathrm{Hess}(N, I)$ is obtained as the intersection of $\mathrm{Hess}(N, I)$ and the Schubert cells of G/B. Let $\Delta = \{\alpha_1, \dots, \alpha_n\}$ be the set of all simple roots of G.

Theorem 2.1 ([5, Theorem 4.10 and Proposition 3.7]). *Regular nilpotent Hessenberg varieties have affine pavings as follows:*

$$\mathrm{Hess}(N, I) = \bigsqcup_{w \in W(I)} (\mathrm{Hess}(N, I) \cap C_w),$$

where $W(I) = \{w \in W \mid w^{-1}(\Delta) \subset \Phi^+ \cup (-I)\}$. *Moreover, when* $\mathrm{Hess}(N, I) \cap C_w$ *is not empty,* $\dim_{\mathbb{C}}(\mathrm{Hess}(N, I) \cap C_w) = \#(N(w) \cap I)$.

The torus action does not preserve $\mathrm{Hess}(N, I)$ but a circle S in T acts on $\mathrm{Hess}(N, I)$ as the restrictoin of the T-action. Recall that a root is a linear homomorphism from $\mathrm{Lie}(T)$ to $\mathbb{R}$. The circle S has a tangent vector v such that it satisfies $\alpha_i(v) = 1$ for any simple root α_i. The cells $\mathrm{Hess}(N, I) \cap C_w$ are invariant under the S-action. Moreover $\mathrm{Hess}(N, I)^S$ is contained in $(G/B)^T$, so we regard the fixed point set as a subset $W(I)$ of the Weyl group.

By Theorem 2.1 we can easily compute the Betti numbers of a regular nilpotent Hessenberg variety. Moreover we proved that the rational cohomology ring of regular nilpotent Hessenberg variety is a quotient ring of $H^*(BT)$ by an ideal generated by a regular sequence of n elements, so it is a Poincaré duality algebra although a regular nilpotent Hessenberg variety is not smooth in general.

3. Weyl type subsets and hyperplane arrangements

In this section we discuss a relation between a hyperplane arrangement associated with the lower ideal I and the regular nilpotent Hessenberg variety $\mathrm{Hess}(N, I)$. The hyperplane arrangement is $\mathcal{A}_I = \{H_\alpha \mid \alpha \in I\}$ in $\mathrm{Lie}(T)$. It is a subarrangement of the Weyl arrangment $\{H_\alpha \mid \alpha \in \Phi^+\}$. The key which connects $\mathcal{A}_I$ and $\mathrm{Hess}(N, I)$ is the chambers of $\mathcal{A}_I$, that is, the connected components of $\mathrm{Lie}(T) \setminus \mathcal{A}_I$.

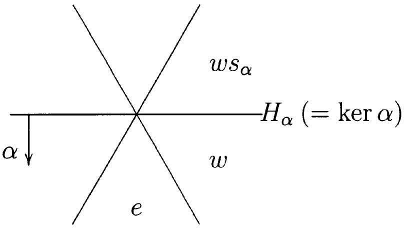
Before discussing hyperplane arrangements, let us introduce the notion of Weyl type subsets of I which was defined in [6]. Let $I \subset \Phi^+$ be a lower ideal. A subset $Y \subset I$ is said to be of **Weyl type** if $\alpha, \beta \in Y$ and $\alpha + \beta \in I$, then $\alpha + \beta \in Y$, and if $\gamma, \delta \in I \setminus Y$ and $\gamma + \delta \in I$, then $\gamma + \delta \in I \setminus Y$. Let $\mathcal{W}^I$ denote the set of the Weyl type subsets of I.

In our paper, we only mentioned that the set of all Weyl type subsets of I can be identified with the fixed point set $\mathrm{Hess}(N, I)^S$ as follows.

Theorem 3.1 ([6]). *Let I be a lower ideal. The map* $\eta : \mathrm{Hess}(N, I)^S \to \mathcal{W}^I$ *defined by* $\eta(w) = N(w) \cap I$ *is a bijection.*

However we can also read the complex dimension of the cell $\mathrm{Hess}(N, I) \cap C_w$ from the corresponding Weyl type subset. It is nothing but $\#\eta(w)$. This is also equal to $\#(N(w) \cap I)$ and is a very natural analogue of the flag varieties. At first glance some readers may think that Weyl type subsets are too formal, so I translate the language of Weyl type subsets to that of hyperplane arrangements which occur naturally from lower ideals.

For translation, we need to verify the connection between the Weyl group and the Weyl chambers. We identify the Weyl group W and the set $C(\mathcal{A}_{\Phi^+})$ of the Weyl chambers as follows. First, identify the unit element e of the Weyl group with the fundamental Weyl chamber (FWC). Second, when two chambers are mirror images with respect to a hyperplane H_α and one has already been identified with $w \in W$, the other is identified with ws_α, where s_α is the reflection for α. Finally, we obtain one to one corresponding between W and $C(\mathcal{A}_{\Phi^+})$ by the iteration of the second rule. It is well-known that this Weyl group action on $C(\mathcal{A}_{\Phi^+})$ is transitive and effective.

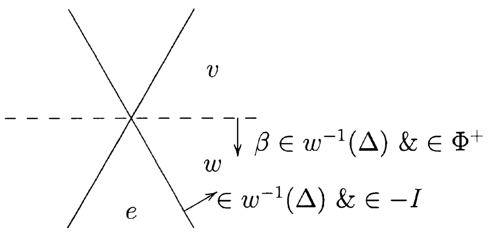


Let J be a subset of Φ^+. For a chamber C of $\mathcal{A}_J$, we obtain two subsets of J: one is the subset Y of all positive roots which evaluate C positively and the other $J \setminus Y$ is that of all positive roots which evaluate C negatively. Conversely a precise pair of two subsets of J determine a chamber of $\mathcal{A}_J$. A Weyl type subset Y of I is a subset of I, so it is natural to try to connect it with a chamber of $\mathcal{A}_I$. By the definition of Weyl type subsets, Y and $I \setminus Y$ are a precise pair. Indeed, Sommers and Tymoczko proved the following proposition.

Proposition 3.2 ([6, Proposition 6.1]). *Let $Y \in \mathcal{W}^I$. There exists $w \in W$ such that $Y = N(w) \cap I$.*

They said noting about hyperplane arrangements in [6], but Proposition 3.2 says that Y and $I \setminus Y$ are a precise pair and they determine the chamber of $\mathcal{A}_I$ containing the Weyl chamber w. Recall that the positive root evaluates the fundamental Weyl chamber positively and that a root contained in $N(w)$ is a root which evaluates the Weyl chamber w negatively. The dimension of the cell $\mathrm{Hess}(N, I) \cap C_w$ is equal to $\#(N(w) \cap I)$, and then it is described as the number of hyperplanes which separate the two chambers of $\mathcal{A}_I$: one is the chamber which contains the fundamental Weyl chamber and the other is the chamber which contains the Weyl chamber w.

The condition $w^{-1}(\Delta) \subset \Phi^+ \cup (-I)$ is translated as follows. Recall that, when α is a simple root, H_α is a wall of the fundamental Weyl chamber, and vice versa. Because the mirror image in Lie(T) denotes the right multiplication of the corresponding reflection in W, $w^{-1}(\Delta)$ denotes the set of the roots whose hyperplanes are walls of the Weyl chamber w. Moreover, for any root $\alpha \in w^{-1}(\Delta)$, α evaluates w positively. Therefore, if w satiesfies the condition $w^{-1}(\Delta) \subset \Phi^+ \cup (-I)$, then w is minimal in the chamber of $\mathcal{A}_I$ with respect to the Bruhat order. By [6, Proposition 6.2], such w is minimum in the chamber. The following picture is a hyperplane arrangement $\mathcal{A}_I$ for $I \setminus \{\beta\}$, where β is a maximal element of I. If v is a fixed point of Hess(N, I) and is not a fixed point of Hess$(N, I \setminus \{\beta\})$, namely $v \in W(I) \setminus W(I \setminus \{\beta\})$, then $v^{-1}(\Delta)$ containes $-\beta$, namely $w\beta \in -\Delta$.



4. Cohomology rings

In this note, we consider the cohomology rings with rational coefficients. Recall that Hess$(N, I) \subset G/B$, so we have the composition of natural maps Hess$(N, I) \to G/B \to BT$. We proved that the induced homomorphism $\varphi_I \colon H^*(BT) \to H^*(\mathrm{Hess}(N, I))$ is surjective as a part of our main theorems, however we admit that this induced homomorphism is surjective without proof here for simplicity. Then our aim is to determine the kernel of this homomorphism.

For analyzing the cohomology ring of some space with a good torus (in this case a circle) action, it is useful to consider the equivariant cohomology ring. By the localization theorem, we have an injective homomorphism

$$H_S^*(\mathrm{Hess}(N, I)) \to H_S^*(\mathrm{Hess}(N, I)^S) \cong \bigoplus_{w \in W(I)} H^*(BS),$$

where $W(I) = \{w \in W \mid w^{-1}(\Delta) \subset \Phi^+ \cup (-I)\}$. The target of the homomorphism is the direct summand of copies of the polynomial ring with one generator, so we can easily calculate which elements of $H_S^*(BT)$ are in the kernel. Let $t \in \mathrm{Lie}(S)^*$ be the dual basis of the tangent vector $v \in \mathrm{Lie}(S)$ defined in Section 2. Of course t is a generater of $H^*(BS)$. Before considering concrete descriptions of elements of $H_S^*(\mathrm{Hess}(N, I))$ in $\bigoplus_{w \in W(I)} H^*(BS)$, let us consider T-equivariant cohomology ring of the flag variety with the following diagram.

$$\begin{array}{ccccc} H_T^*(BT) & \longrightarrow & H_T^*(G/B) & \longrightarrow & H_T^*((G/B)^T) \cong \bigoplus_W H^*(BT) \\ \downarrow & & \downarrow & & \downarrow \\ H_S^*(BT) & \longrightarrow & H_S^*(G/B) & \longrightarrow & H_S^*((G/B)^S) \cong \bigoplus_W H^*(BS) \\ & & \downarrow & & \downarrow \\ & & H_S^*(\mathrm{Hess}(N,I)) & \longrightarrow & H_S^*(\mathrm{Hess}(N,I)^S) \cong \bigoplus_{W(I)} H^*(BS), \end{array}$$

where all homomorphisms except for left horizontal arrows are induced by the inclusion maps and the homomorphism $H^*(BT) \cong \mathbb{Q}[\alpha_1, \dots, \alpha_n] \to H^*(BS)$ induced from $S \hookrightarrow T$ is given by $\alpha_i \mapsto t$ for any i. For a root α of G, let L_α be the complex line bundle $ET \times_T \mathbb{C}_\alpha \to BT$. It is a T-equivariant bundle with the left multiplication of T. For a root α of G, we regard it as the equivariant Euler class $e^T(L_\alpha)$ in $H_T^*(BT)$ or the S-equivariant Euler one $e^S(L_\alpha)$ in $H_S^*(BT)$ and then an element of $H_T^*(G/B)$ or $H_S^*(G/B)$ under the left horizontal homomorphisms. Moreover we regard it as the non-equivariant one when we consider the ordinary cohomology rings. Then the w-component of the image of α under $H_T^*(BT) \to H_T^*((G/B)^T) \cong \bigoplus_W H^*(BT)$ is $w\alpha$. On the other hand, the equivariant parameters in $H_T^*(BT)$ are mapped to diagonal elements in $H_T^*((G/B)^T)$. Put $\mathfrak{n}_S(I) = \ker\left(H_S^*(BT) \to H_S^*(\mathrm{Hess}(N,I))\right)$.

Proposition 4.1. *When I is a lower ideal and $\alpha \in I$ is maximal, $n_S(I \setminus \{\alpha\}) \subset \mathfrak{n}_S(I) : (\alpha + t)$.*

Proof. According to the localization theorem, $H_S^*(\mathrm{Hess}(N,I)) \to H_S^*(\mathrm{Hess}(N,I)^S) \cong \bigoplus_{W(I)} H^*(BS)$ is injective. Hence the condition $g \in \mathfrak{n}_S(I)$ means that its image in $\bigoplus_{W(I)} H^*(BS)$ vanishes. For $v \in W(I) \setminus W(I \setminus \{\alpha\})$, the v-component of $\alpha + t \in H_S^*(BT)$ is $(\alpha + t)_v = -t + t = 0$ because $-\alpha \in v^{-1}(\Delta)$ and $v\alpha$ is mapped to t. Hence $\alpha + t$ vanishes on $W(I) \setminus W(I \setminus \{\alpha\})$. $\square$

Let $\mathfrak{n}(I)$ be the image of $\mathfrak{n}_S(I)$ under $H_S^*(\mathrm{Hess}(N,I)) \to H^*(\mathrm{Hess}(N,I))$. We proved the following lemma.

Lemma 4.2 ([3, Lemma 5.5])**.** *If φ_I is surjective, then $\mathfrak{n}(I)$ agrees with the kernel of φ_I.*

Thanks to the equivariant description, we can detect $\mathfrak{n}(I)$ concretely. From Proposition 4.1 and the following commutative diagram, we obtain $n(I \setminus \{\alpha\}) \subset \mathfrak{n}(I) \colon \alpha$.

Let $\beta_I = \prod_{\alpha \in \Phi^+ \setminus I} \alpha$. Then $\mathfrak{n}(I) \subset n(\Phi^+) \colon \beta_I = (H^{>0}(BT)^W) \colon \beta_I$.

$$\begin{tikzcd}
H_S^*(BT) \arrow[rr] \arrow[dr] \arrow[ddd] & & H_S^*(\mathrm{Hess}(N, I \setminus \{\alpha\})) \arrow[ddd] \\
& H_S^*(\mathrm{Hess}(N, I)) \arrow[ur] \arrow[d] & \\
& H^*(\mathrm{Hess}(N, I)) \arrow[dr] & \\
H^*(BT) \arrow[rr] \arrow[ur, "\varphi_I"'] & & H^*(\mathrm{Hess}(N, I \setminus \{\alpha\}))
\end{tikzcd}$$

Note that the Poincaré series of $\mathrm{Hess}(N, I)$ is palindromic because of the existence of the antipodal chamber for each chamber in $\mathcal{A}_I$. We proved $n(\Phi^+) \colon \beta_I \subset \mathfrak{n}(I)$ also [3, Theorem 7.1], comparing the Poincaré series of $\mathrm{Hess}(N, I)$ with that of a quotient ring of $H^*(BT)$ by a ideal $\mathfrak{a}(I)$ which comes from the logarithmic derivation module of $\mathcal{A}_I$. It is known that $H^*(BT)/\mathfrak{a}(I)$ is a Poincaré duality algebra [1, Theorem 1.1] and that $H^*(BT)/\mathfrak{n}(I)$ and $H^*(BT)/\mathfrak{a}(I)$ have the same socle degree [7, Theorem 1.1] and the same top degree. By the above facts, $\mathfrak{n}(I)$ and $\mathfrak{a}(I)$ must coincide. Therefore the ideal $\mathfrak{n}(I)$ is generated by a regular sequence with n elements of $H^*(BT)$. Such sequence is known when G is of type A [2] and we found it when G is of type B, C, G_2 [3, Section 10]. However, when G is of other type, the problem to find such regular sequences is still open.

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